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STUDY OF SYSTEMS OF DIFFERENTIAL EQUATIONS NEAR THE BOUNDARIES OF THE STABILITY REGION

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Necessary and sufficient conditions of applicability of the Magnus method to study of the systems of nonlinear differential equations near the boundaries of the stability region are given.

Magnus in [1] extended the Krylov — Bogoliubov method to the case on nonlinear systems near the boundaries of their region of stability. The essence of the method was, that the nonlinear system

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} a_{ij}x_j + \mu_i f_i(x_1, x_2, \ldots, x_n) \tag{1}$$

the characteristic equation of which has nor more than one pair of pure imaginary roots and where the nonlinear function f_i is expanded into a Fourier series and replaced by the linear system

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} \left(\bar{a}_{ij}^* \frac{dx_i}{dt} + \bar{a}_{ij}x_j\right) \tag{2}$$

The solution is sought in the form

$$x_j = A_j \sin\psi_j, \quad \psi_j = \omega t + \varphi_j \tag{3}$$

Substituting (3) into (1) and (2) under the assumption that all oscillations have the same frequency ω but different amplitudes A_j and phases φ_j, we find that f_i are $2\pi/\omega$ -periodic functions. Expanding these functions into Fourier series and retaining the first order harmonies only, we obtain

$$f_i = a_{i1}\cos\psi_1 + b_{i1}\sin\psi_1$$

$$\begin{Bmatrix} a_{i1} \\ b_{i1} \end{Bmatrix} = \frac{1}{\pi}\int_0^{2\pi} f_i(AK_1\sin\psi_1, \ldots, AK_n\sin\psi_n)\begin{Bmatrix}\cos \\ \sin\end{Bmatrix}\psi_1\, d\psi_1, \quad A_j = AK_j$$

Substituting f_i into (1) and (2) we obtain, respectively,

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} a_{ij}AK_j\sin\psi_1 + \mu_i(a_{i1}\cos\psi_1 + b_{i1}\sin\psi_1) \tag{4}$$

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} (\bar{a}_{ij}^* AK_j\omega\cos\psi_j + \bar{a}_{ij}AK_j\sin\psi_j) \tag{5}$$

Equating the coefficients accompanying the like harmonics in the systems (4) and (5), we obtain

$$\bar{a}_{ij} = a_{ij}, \quad \bar{a}_{ij}^* = 0 \quad \text{for } j \neq 1$$

$$\bar{a}_{ij} = a_{i1} + \frac{\mu}{AK_1} b_{i1}, \quad \bar{a}_{ij}^* = \frac{\mu}{A\omega K_1} a_{i1} \quad \text{for } j = 1$$

Clearly, when $\mu_i = 0$ the systems (1) and (2) become identically equal and at sufficiently small μ_i they approach each other sufficiently closely. Applying the Hurritz criterion to system (2), we find the boundary at stability. Since the parameters of the system (2) depend on the amplitude A, therefore according to Magnus the character of the boundary of stability of the system (2) and hence also of the system (1), provided that μ_i are sufficiently small, is determined by the relative position of the boundary of stability $R = 0$ and of the A - curve determined by the function $\bar{a}_{ij} = a(A)$ and $\bar{a}_{ij}^* = a^*(A)$, defined parametrically in the $\bar{a}_{ij}, \bar{a}_{ij}^*$ parameter space.

The values of $A = A_i$ corresponding to the points of intersection of the boundary of stability with the A -curve, represent the first order approximations to the amplitudes of the steady oscillations. Magnus asserts that the boundary $R = 0$ is safe if $dR / dA_{A=0} > 0$ and unsafe if $dR / dA_{A=0} < 0$. Bautin generalizes the Liapunov's method to show in [2] that the boundary of the region of stability is safe when the Liapunov's coefficient $\alpha_3 = L(\lambda_0) < 0$, and unsafe when $L(\lambda_0) > 0$. Clearly, the point separating the safe and unsafe parts of the boundary of the region of stability found, using the Magnus method, need not coincide with the point obtained in [2].

Thus the conditions formulated by Magnus are necessary and sufficient, provided that the following theorem holds: if at some point M_0 of the parameter space of the system (1) the Liapunov's coefficient $\alpha_1 = R = 0$ and $\alpha_3 = L(\lambda_0) < 0$ $(\alpha_3 > 0)$, then the character of stability in the system (1), with the values of the parameters corresponding to the point M_0, remains undistorted when the Magnus method is used in its investigation, if and only if $dR / dA > 0$ $(dR / dA < 0)$.

P r o o f. N e c e s s i t y. Let the boundary of the region of stability be safe at the point M_0; i.e. $\alpha_3 = L(\lambda_0) < 0$. We shall show that the inequality $dR / dA > 0$ must hold. Assume the opposite, i.e. that $dR / dA < 0$. Magnus has shown that in this case the A -curve originating within the region of stability intersects the boundary of stability at some $A = A_0$ corresponding to the amplitude of unstable oscillation, and this contradicts the results obtained in [2] according to which a stable limiting cycle exists at $L(\lambda_0) < 0$ corresponding to stable oscillations.

S u f f i c i e n c y. Let the boundary be unsafe at the point M_0. We shall prove that for $dR / dA > 0$, the Magnus method does not distort the character of the boundary at this point. Indeed, when $dR / dA > 0$, the A -curve originating within the region of stability intersects the boundary of stability at some point $A = A_1$ corresponding to the amplitude of a stable oscillation and this agrees with the results of [2]. This implies that when the condtions of the theorem do not hold, the Magnus method yields an incorrect result.

The case when the boundary of the region of stability is unsafe at the point M_0, can be proved in a similar manner.

Example. Let us consider the system

$$dx / dt = y, \ dy / dt = -x + dy + b_{21}x^2y + b_{03}y^3$$

We have for this system $\alpha_3 = {}^1/_4 \pi c, c = -(b_{21} + 3\,b_{03})$, i.e. $\alpha_3 < 0$ for $c > 0$. Therefore the boundary of stability the points of which satisfy the conditions $d = 0$ and $c > 0$, is safe. Applying the Magnus method, we obtain the linear system

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -x + \bar{a}^*y, \quad \bar{a}^* = d + \frac{1}{4}cA^2$$

for which we have

$$\frac{dR}{dA} = \frac{d}{dA}\left(d + \frac{1}{4}cA^2\right) = \frac{1}{2}cA > 0$$

since $c > 0$ by assumption and the conditions of the theorem hold. We can take $A = \sqrt{4\,|\,d\,/\,c\,|}$ as the approximate value of the amplitude.

REFERENCES

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